

Finite temperature properties of quantum Lifshitz transitions between valence bond solid phases: An example of ‘local’ quantum criticality

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We study the finite temperature properties of quantum magnets close to a continuous quantum phase transition between two distinct valence bond solid phases in two spatial dimension. Previous work has shown that such a second order quantum ‘Lifshitz’ transition is described by a free field theory and is hence tractable, but is nevertheless non-trivial. At $T > 0$, we show that while correlation functions of certain operators exhibit ω/T scaling, they do not show analogous scaling in space. In particular, in the scaling limit, all such correlators are purely *local* in space, although the same correlators at $T = 0$ decay as a power law. This provides a valuable microscopic example of a certain kind of ‘local’ quantum criticality. The local form of the correlations arise from the large density of soft modes present near the transition that are excited by temperature. We calculate exactly the autocorrelation function for such operators in the scaling limit. Going beyond the scaling limit by including irrelevant operators leads to finite spatial correlations which are also obtained.

I. INTRODUCTION

Recent theoretical work¹ has discussed strange and unusual phenomena in the vicinity of certain quantum phase transitions in insulating magnets and related systems. These phenomena -dubbed ‘deconfined quantum criticality’ - do not fit in easily into the Landau-Ginzburg-Wilson paradigm for phase transitions. Such quantum critical points seem to be most aptly described in terms of fractionalized degrees of freedom that interact through emergent gauge interactions. These fractionalized modes are in general absent in the phases on either side of the transition but rear their head right at the critical point.

Perhaps the most analytically tractable example of such a deconfined quantum critical point (or indeed of *any* interesting quantum critical point in dimensions bigger than one) is provided by a phase transition between two different valence bond solid phases of spin-1/2 quantum Heisenberg magnets that was discussed in Ref. 2,3. A specific example of such a quantum ‘Lifshitz’ transition is a phase transition between a featureless valence bond solid and a different translation broken one on a bilayer honeycomb lattice. Closely related are the better studied transitions that occur at the Rokhsar-Kivelson(RK) points⁴ of quantum dimer models on bipartite lattices. For instance on the square or honeycomb lattices, the RK point separates two ordered conventional phases. Unfortunately the RK point corresponds to a special very fine tuned multicritical point. (This difficulty is however not present for the bilayer honeycomb model). The tractability of these quantum phase transitions arises from the existence of a free field description. Despite this the theory has non-trivial structure as shown in Refs. 2,3,5. A close mathematical analogy may be drawn with non-trivial critical points in 1 + 1 dimensions which too have free field descriptions, and are hence tractable. The free field description of the two dimensional critical points of interest in this paper may be given either in terms

of a (non-compact) $U(1)$ gauge theory or equivalently in terms of its dual sine-Gordon theory. The latter may be more familiar to readers conversant with ‘height’ descriptions of quantum dimer models^{6,7} and will be used through out this paper. In this height description the critical theory takes the form:

$$S_0 = \frac{1}{2} \int d\tau d^2x \{ (\partial_\tau \chi)^2 + K (\nabla^2 \chi)^2 \} \quad (1)$$

Here χ is the height field. This theory actually describes a fixed line that is parameterized by K ⁸. Further details may be found in Ref. 2,3,5. We first note that the form of the critical action immediately implies that the dynamic critical exponent $z = 2$. Various physical observable have non-trivial scaling structure at this critical point at zero temperature. The purpose of the present paper is to focus on finite temperature correlators of this critical theory - in particular for dynamical correlators. The free field form makes these calculations feasible - a unique property of this quantum transition among other non-trivial two dimensional ones.

Of particular interest to us will be operators such as $\exp(2i\pi\chi)$. Such operators correspond in the gauge theory interpretation to monopole or instanton events which change the total gauge flux of a state by 2π . In the context of quantum dimer models at their RK points, these operators have another interpretation⁶. They are simply the order parameters for one of the phases (the columnar/plaquette) on one side of the transition. As expected, their correlators decay as power laws in both space and time at zero temperature. In this paper we calculate the correlators of such operators at finite temperature. Remarkably in the scaling limit we show that these have the striking property of being *short ranged* in space. The autocorrelation (at two different times at the same spatial point) is however non-zero and non-trivial. This remarkable property implies that at a non-zero temperature T the corresponding frequency (ω) and momentum (k) dependent susceptibility shows ω/T scaling but has no mo-

mentum dependence! We show these features by exact calculation of the universal scaling function for the susceptibility. The unusual structure found in the scaling limit suggests that any spatial structure in the correlator is due to formally irrelevant terms that correct the leading scaling behavior. We determine the form of these corrections to scaling that restore spatial correlations.

The theoretical phenomena found in this paper share some similarities with various ideas that have been proposed in theories of many interesting correlated materials. Indeed ω/T scaling is seen in experiments on a number of different systems - most prominently in the normal state of optimally doped cuprates and in the non-fermi liquid metals near heavy electron critical points. In general, evidence for scaling phenomena in spatial correlations is much weaker (for instance in the cuprates). An interesting viewpoint on such scaling in various strange metals is to attribute it to universal singularities of some proximate critical point. A number of workers have advocated some kind of spatially localized fluctuations at the relevant quantum critical points to account for the weaker signatures of scaling in spatial correlations^{9,10}. However the theoretical framework for description of such exotic critical points (should they even exist) is unclear at present. The model described in the present paper may perhaps shed some light on such theoretical issues (see Section V).

II. BRIEF REVIEW OF ZERO TEMPERATURE CRITICALITY AND LATTICE EFFECTS

The critical theory associated with the zero temperature Lifshitz transition between VBS phases has been discussed in detail elsewhere^{2,3}. Here we simply note that under the appropriate conditions (eg. a bilayer honeycomb spin system with antiferromagnetic interlayer exchange) a continuous transition is possible, from a VBS state with zero ‘tilt’ to one where the tilt begins to grow. While the details of the physics on the tilted side is complicated², here we will restrict ourselves to reviewing briefly properties of the zero temperature transition itself. It is convenient to use the height representation of the VBS states, in terms of which the effective action (eg. for the bilayer honeycomb model) takes the form:

$$S = S_0 + S_1 + S_{inst} \quad (2)$$

$$S_0 = \frac{1}{2} \int d^2x d\tau \{ (\partial_\tau \chi)^2 + \rho (\nabla \chi)^2 + K (\nabla^2 \chi)^2 \} \quad (3)$$

$$S_1 = \int d^2x d\tau \frac{u}{4} |\Delta \chi|^4 + \dots \quad (4)$$

$$S_{inst} = - \int d^2x d\tau \lambda \cos(2\pi \chi) \quad (5)$$

where the height field χ plays the role of a dual gauge potential and is related to the electric field strengths of the gauge theoretic description via $E_i = \epsilon_{ij} \partial_j \chi$. In this interpretation the λ term corresponds to monopole tun-

neling events, and the transition of interests occurs as we tune the ρ term through zero. At the critical point, the u term is marginally irrelevant, and the monopole tunneling events are also irrelevant for some range of K ($0 < K < (\frac{\pi}{8})^2$). The critical action in the scaling limit is simply given by:

$$S_c = \frac{1}{2} \int d^2x d\tau \{ (\partial_\tau \chi)^2 + K (\nabla^2 \chi)^2 \} \quad (6)$$

In addition there are logarithmic corrections to correlation functions arising from the marginally irrelevant term. Although this critical action is Gaussian, there exist operators in this theory that have nontrivial scaling dimensions. In particular, we will focus on the monopole insertion operator $V^\dagger(r, \tau) = e^{2\pi i \chi(r, \tau)}$. It may be easily seen that in the scaling limit, correlations ($C_0(r, \tau) = \langle \mathcal{T}_\tau V^\dagger(r, \tau) V(0, 0) \rangle$) of this operator have a non-trivial power law behaviour:

$$C_0(r, 0) \sim r^{-\frac{\pi}{\sqrt{K}}} \quad (7)$$

$$C_0(0, \tau) \sim \tau^{-\frac{\pi}{2\sqrt{K}}} \quad (8)$$

In fact the correlation function can be written in the scaling form $C_0(r, \tau) \sim r^{-\frac{\pi}{\sqrt{K}}} f(r^2/\tau)$, which is a result of the dynamic critical exponent $z = 2$ in this theory. For completeness we note that the scaling function is found to be:

$$f(z) = e^{-\frac{\pi}{2\sqrt{K}} \Gamma(0, z)} \quad (9)$$

$$\Gamma(0, z) = \int_z^\infty dt \frac{e^{-t}}{t} \quad (10)$$

We now consider the form of these correlators at finite but small temperatures, where the physics is expected to be controlled by the zero temperature quantum critical point.

III. CALCULATION OF CORRELATIONS AT FINITE TEMPERATURES

Before proceeding to an explicit calculation of the correlators at finite temperature, we first note what we might naively expect such a calculation to produce. Consider correlations of monopole insertion operators at a finite temperature $T > 0$ ($\tilde{C}_T(r, \tau)$). In the quantum critical regime the inverse temperature just provides a cutoff in imaginary time and is also expected to induce a spatial length scale below which correlations look quantum critical. Moreover, one may expect that for an operator with a nontrivial scaling dimension, finite temperature scaling forms for the correlation function, in frequency space, may be written as:

$$\tilde{C}_T(r, \omega) = \frac{1}{T} r^{-\frac{\pi}{\sqrt{K}}} F(r^2 T, \omega/T) \quad (11)$$

In particular, the autocorrelation function may be expected to have the scaling form:

$$\tilde{C}_T(0, \omega) = \frac{1}{T} \omega^{\frac{\pi}{2\sqrt{K}}} g_1(\omega/T) \quad (12)$$

while the equal time correlation function should obey:

$$C_T(r, \tau = 0) = r^{-\frac{\pi}{\sqrt{K}}} g_2(r\sqrt{T}) \quad (13)$$

Further one might expect logarithmic violations of these scaling forms if the marginally irrelevant quartic term is allowed in the microscopic model. Below we will show that while the autocorrelation function does exhibit the ω/T scaling form shown in Eqn. (15), the equal time correlation function at spatially separated points does *not* exhibit the expected scaling form of Eqn. (13)! Moreover, in the scaling limit the correlation function in Eqn. (13) vanishes at any two spatially separated points due to an infrared divergence. In other words the correlations in the scaling limit are purely local. This is demonstrated in the two sections below. In order to obtain a non-vanishing part for the spatial correlators we have to go beyond the scaling limit and include the effect of operators that are irrelevant at the zero temperature critical point. In situations in which the quartic u term is allowed its marginal irrelevance leads to logarithmic violations of scaling. We will show that non-vanishing spatial correlations result entirely from these logarithmic corrections to the naive scaling limit. These effects, as well as the effects arising from gapped spinons will be discussed in the third section below.

A. Calculations in the Scaling Limit

1. Autocorrelation Function

In this subsection we calculate the autocorrelation function at finite temperatures of the monopole insertion operator. The spectral function corresponding to this autocorrelation function in the scaling limit can be calculated exactly and is displayed in Eqn. (23). In the following we describe the details of that calculation.

To calculate finite temperature properties near the quantum critical point in the scaling limit, we use the fixed point Euclidean action:

$$S_c = \int_0^{\frac{1}{T}} d\tau \int d^2x \frac{1}{2} \{ (\partial_\tau \chi)^2 + K (\nabla^2 \chi)^2 \} \quad (14)$$

where T is the temperature. The free field nature of this action allows us to readily compute correlation functions. The marginally irrelevant quartic term has been dropped in this subsection - that will only lead to logarithmic corrections to the results derived below, and hence will be ignored in what follows. However it will play an important role in the structure of the spatial correlations and will be reinstated in Section III B 1.

We begin with a calculation of the autocorrelation function in the scaling limit at non-zero temperature. This is defined to be

$$C_0(0, \tau) = \langle e^{i2\pi\chi_0(\tau)} e^{-i2\pi\chi_0(0)} \rangle \quad (15)$$

For the Gaussian action this is readily evaluated at finite temperature and takes the form:

$$C_T(0, \tau) = \exp(-T \sum_{n=-\infty}^{\infty} (1 - e^{i\omega_n \tau}) \int \frac{d^2q}{\omega_n^2 + Kq^4}) \quad (16)$$

After integrating over \vec{q} we get:

$$C_T(0, \tau) = \exp(-\frac{\pi}{2\sqrt{K}} \sum_{n=1}^{\infty} \frac{1 - \cos(2\pi n \tau T)}{n}) \quad (17)$$

Performing the sum over n we have¹⁴:

$$C_T(0, \tau) = \frac{c_0 T^\eta}{(1 - \cos(2\pi \tau T))^{\frac{\eta}{2}}} \quad (18)$$

where

$$\eta = \frac{\pi}{2\sqrt{K}} \quad (19)$$

and c_0 is a constant. Clearly, in the zero temperature limit this reduces to equation (15), while at any finite temperature it has a scaling form (i.e. $T^{-\eta} C_T(0, \tau)$ is only a function of the product $T\tau$). Thus, the autocorrelation function exhibits the usual scaling form.

We will proceed below to explicitly calculate the finite temperature spectral function associated with this correlator, since it is one of the few situations in 2+1 dimensional criticality where this may be done. We will follow closely an almost identical calculation of the finite temperature spectral function of a one dimensional Luttinger liquid in Ref.¹¹. We begin by taking the Fourier transform of this imaginary time auto correlation function:

$$\tilde{C}_T(0, i\omega_n) = \frac{c_0 T^{\eta-1} \Gamma(\frac{1-\eta}{2}) \Gamma(\frac{\eta}{2} + \frac{|\omega_n|}{2\pi T})}{\Gamma(\frac{\eta}{2}) \Gamma(1 - (\frac{\eta}{2} - \frac{|\omega_n|}{2\pi T}))} \quad (20)$$

Analytic continuation to real frequencies is now easily performed. In order to obtain the retarded Green's function we need to approach the real frequency axis from the upper half plane, i.e. perform the replacement $|\omega_n| \rightarrow -i\omega$. This yields:

$$\tilde{C}_T^{\text{Ret}}(0, \omega) = \frac{c_0 T^{\eta-1} \Gamma(\frac{1-\eta}{2}) |\Gamma(\frac{\eta}{2} - \frac{i\omega}{2\pi T})|^2}{2^{\eta/2} \pi^{3/2} \Gamma(\frac{\eta}{2})} \sin(\frac{\pi\eta}{2} + \frac{i\omega}{2T}) \quad (21)$$

where we have used the well known identity $1/\Gamma(\frac{1}{2} - z) = \frac{1}{\pi} \Gamma(\frac{1}{2} + z) \cos \pi z$. The spectral function associated with this autocorrelator is then given by:

$$A_T(\omega) = 2 \text{Im}(\tilde{C}_T^{\text{Ret}}(0, \omega)) \quad (22)$$

$$= c_0 T^{\eta-1} \frac{\sinh(\frac{\omega}{2T}) |\Gamma(\frac{\eta}{2} - \frac{i\omega}{2\pi T})|^2}{2^{\frac{\eta-2}{2}} \sqrt{\pi} \Gamma(\frac{\eta}{2}) \Gamma(\frac{\eta+1}{2})} \quad (23)$$

We note that the above spectral function¹⁵ is an odd function of ω and writing it in the form:

$$A_T(\omega) = c_0 T^{\eta-1} F(\omega/T) \quad (24)$$

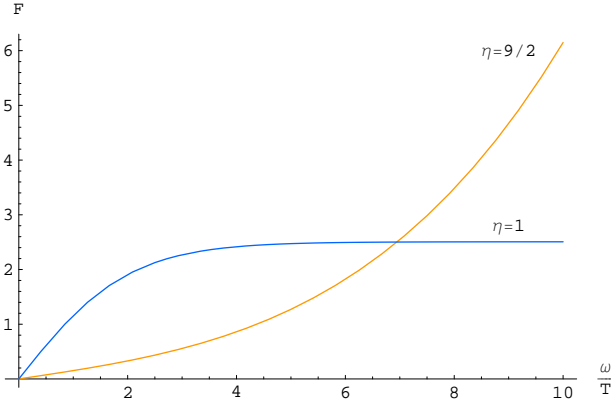


FIG. 1: The scaling part of the spectral function, defined in Eqn (24) associated with the monopole operator autocorrelation function plotted for two different values of K (and hence η , the scaling dimension of the monopole operator). The value $\eta = 1$ corresponds to the scaling dimension of the monopole operator at the square lattice RK point. The value $\eta = 9/2$ corresponds to a point on the stable fixed line that controls the transition between VBS phases of a bilayer honeycomb quantum magnet.

we see that it satisfies ω/T scaling.

The asymptotic behaviours of the scaling function are as follows. For small frequencies, the function must be analytic at finite temperatures and hence $F(x) \sim x$ for $x \ll 1$ while at high frequencies we find $F(x) \sim x^{\eta-1}$ which yield the expected zero temperature form of the spectral function $A(\omega) \sim \omega^{\eta-1}$. Finally we note that for $\eta = 1$ which is the exponent relevant for the square lattice RK point, the scaling function takes on a particularly simple form:

$$F\left(\frac{\omega}{T}\right) = \sqrt{\frac{8}{\pi}} \tanh\left(\frac{\omega}{2T}\right) \quad (25)$$

This, along with the scaling function for $\eta = 9/2$ (which corresponds to a value of $K = \pi^2/81$ that is on the stable fixed line for the bilayer honeycomb model) is plotted in Fig. III A 1.

Thus the free Gaussian action has given us a unique opportunity to extract exact information on the non-trivial real time dynamics at non-zero temperatures above a 2+1 dimensional quantum critical point.

2. Equal Time Spatial Correlations

We now proceed to calculate the finite temperature, equal-time correlator of the monopole insertion operator:

$$C_T(r, 0) = \langle e^{i2\pi\chi_r(0)} e^{-i2\pi\chi_0(0)} \rangle_T \quad (26)$$

Using the form (14) for the action we get the following expression for the correlator:

$$C_T(r, 0) = \exp \left\{ -T \int d^2q \sum_{n=-\infty}^{\infty} \frac{(1 - e^{i\vec{q} \cdot \vec{r}})}{\omega_n^2 + Kq^4} \right\} \quad (27)$$

The sum over the Matsubara frequencies ($\omega_n = 2\pi nT$) is easily performed:

$$T \sum_{\omega_n} \frac{1}{\omega_n^2 + Kq^4} = \int \frac{d\omega}{2\pi} \frac{1}{\omega^2 + Kq^4} + \frac{1}{\sqrt{K}q^2} \frac{1}{e^{\frac{\sqrt{K}q^2}{T}} - 1} \quad (28)$$

Putting this back into equation (27), we see that the first term on the right hand side of (28) generates the zero temperature correlation function. Thus:

$$C_T(r, 0) = C_0(r, 0) \Phi_T(r) \quad (29)$$

Where $C_0(r, 0) \sim \frac{1}{r\sqrt{K}}$ is the zero temperature correlator and

$$\Phi_T(r) = \exp\left(-\int \frac{d\theta dq}{\sqrt{K}q} \frac{1 - e^{iqr \cos \theta}}{e^{\frac{\sqrt{K}q^2}{T}} - 1}\right) \quad (30)$$

Note however that the integration over q diverges logarithmically at its lower limit. Introducing an infrared cutoff ($1/L$) which is the inverse of the linear dimension of the system, we have:

$$C_T(r, 0) \sim \frac{1}{r\sqrt{K}} e^{-\frac{\pi}{2K} T r^2 \log L} \rightarrow 0 \quad (L \rightarrow \infty, r \neq 0) \quad (31)$$

Thus, at any finite temperature, correlations at spatially separated points vanish in the thermodynamic limit ($L \rightarrow \infty$). Note, that if the zero temperature limit is taken first, then we recover the zero temperature correlator. Thus, finite temperature correlations (both equal and non-equal time) of this operator in the scaling limit vanish at spatially separated points. In other words the correlations are purely *local*. In fact it is easy to pinpoint the origin of this divergence. The Matsubara sum in (27) is dominated by the zero frequency contribution which implies that we must examine the integral:

$$T \int d^2q \frac{(1 - e^{i\vec{q} \cdot \vec{r}})}{Kq^4}$$

which is clearly logarithmically divergent at small q and hence leads to the result in equation (32). This divergence arises from the large number of soft modes present at low energies in this system, due to the quadratic dispersion. The physical origin of this large number of low lying states is a consequence of the fixed point action (14) being symmetric under arbitrary translations $\chi \rightarrow \chi + \text{const.}$ and arbitrary tilts $\chi \rightarrow \chi + \vec{Q} \cdot \vec{r}$ (for any \vec{Q}) of the height field. Since the scaling limit contribution to the correlation function vanishes at spatially separated points, we need to go beyond the scaling limit in order to obtain a finite contribution.

B. Beyond the Scaling Limit: Effect of Irrelevant Operators and Gapped Spinons

1. Effect of the Quartic Operator

So far we have been concerned with the finite temperature properties of the quantum critical system in the scaling limit, where all operators that are irrelevant at the quantum critical point were omitted. In particular the quartic term, which is marginally irrelevant at the quantum critical point, and leads only to logarithmic corrections to correlators, was dropped. We now discuss the fate of the spatial correlations at non-zero temperature once the quartic term is included. Subsequently we discuss the effect of thermally excited spinons (vortices in the height field). The crucial effect is that a non-zero value is generated for the strength of the quadratic spatial gradient term (denoted ρ in Eqn. 2 above) - this is familiar in the analysis of quantum criticality for models at or above their upper critical dimension¹³. Below we show how this restores spatial correlations for the monopole operators discussed above.

We begin by calculating the value of ρ at low non-zero temperatures above the quantum critical point. This may be done by considering the renormalization group flows at $T \neq 0$ away from the zero temperature critical fixed point. Equivalent results are obtained in a simple approximation that is exact for a suitable large- N generalization of the model. We will present this calculation below.

Retaining the quartic term in our continuum effective action we have:

$$S = \frac{1}{2} \int_0^{\frac{1}{T}} d\tau \int d^2x \{ (\partial_\tau \chi)^2 + \rho^{QC} (\nabla \chi)^2 + K (\nabla^2 \chi)^2 + u (\nabla \chi)^4 \} \quad (32)$$

Note that we have included a bare stiffness term ρ^{QC} which is determined by requiring that at zero temperature the system is at the quantum critical point.

In order to handle the quartic term, we will resort to the large N approximation, where we will assume that $\vec{\chi}$ is an N component field. Of course, we are strictly interested in the $N = 1$ limit, but it will be convenient to consider a large number of components. In order to obtain a sensible action in this limit, we rescale u to u/N so that the quartic term in the action takes the form $\frac{u}{N} |\nabla \vec{\chi}|^4$. We can rewrite the partition function as:

$$Z = \int [D\chi] [D\lambda] e^{-S[\vec{\chi}, \lambda]} \quad (33)$$

where

$$S = \int d\tau d^2x \left[\frac{|\partial_\tau \vec{\chi}|^2}{2} + K \frac{|\nabla^2 \vec{\chi}|^2}{2} + \rho^{QC} \frac{|\nabla \vec{\chi}|^2}{2} + i\lambda \frac{|\nabla \vec{\chi}|^2}{2} + \frac{N\lambda^2}{16u} \right] \quad (34)$$

Now replacing $i\lambda$ with $\rho_{eff} - \rho^{QC}$, (this choice will simplify notation) we have a path integral over the fields $\vec{\chi}$ and ρ_{eff} . Now performing the path integration over $\vec{\chi}$ we obtain:

$$Z \sim \int [D\rho_{eff}] e^{-S[\rho_{eff}]} \quad (35)$$

where:

$$S[\rho_{eff}] = \frac{NL^2}{2} \int \frac{d^2q}{(2\pi)^2} \sum_{\omega_n} \log(\omega_n^2 + Kq^4 + \rho_{eff} q^2) - \frac{N(\rho_{eff} - \rho^{QC})^2}{16u} \beta L^2 \quad (36)$$

here $\omega_n = 2\pi nT$ are Matsubara frequencies. We see that $S[\rho_{eff}]$ is of order N and so in the large N limit, we can perform the integration over ρ_{eff} using *saddle point* method. This gives the following self consistent equation for ρ_{eff} :

$$\begin{aligned} \rho_{eff} - \rho^{QC} &= 4uT \int \frac{d^2q}{(2\pi)^2} \sum_{\omega_n} \frac{q^2}{\omega_n^2 + Kq^4 + \rho_{eff} q^2} \\ &= 4u \int \frac{d^2q}{(2\pi)^2} \frac{q^2}{\sqrt{Kq^4 + \rho_{eff} q^2}} \left(\frac{1}{2} + \frac{1}{e^{\frac{\sqrt{Kq^4 + \rho_{eff} q^2}}{T}} - 1} \right) \end{aligned} \quad (37)$$

where we have used the identity:

$$T \sum_{\omega_n} \frac{1}{\omega_n^2 + a^2} = \frac{1}{a} \left[\frac{1}{2} + \frac{1}{e^{\beta a} - 1} \right]$$

The bare parameter ρ^{QC} is calculated by requiring that at zero temperature the system is at the quantum critical point, i.e. $\rho_{eff} = 0$. Setting $\rho_{eff} \rightarrow 0$ as $T \rightarrow 0$ in equation (37), we get:

$$-\rho^{QC} = 4u \int \frac{d^2q}{(2\pi)^2} \frac{q^2}{2\sqrt{Kq^4}} \quad (38)$$

Putting all these together we get the following self consistent equation in which the only unknown parameter is ρ_{eff} :

$$\begin{aligned} \rho_{eff} &= 4u \int \frac{d^2q}{(2\pi)^2} \left\{ \frac{q^2}{\sqrt{Kq^4 + \rho_{eff} q^2}} \right. \\ &\quad \left. \left(\frac{1}{2} + \frac{1}{e^{\frac{\sqrt{Kq^4 + \rho_{eff} q^2}}{T}} - 1} \right) - \frac{1}{2\sqrt{K}} \right\} \end{aligned} \quad (39)$$

This integral is performed assuming a high momentum cutoff Λ . Assuming we are in the regime where $\rho_{eff} \ll \sqrt{KT}$ we get:

$$\rho_{eff} = \frac{uT}{\pi K} \frac{\log\left(\frac{\sqrt{KT}}{\rho_{eff}}\right)}{1 + \frac{u}{4\pi K^{\frac{3}{2}}} [\log\left(\frac{4K\Lambda^2}{\rho_{eff}}\right) - 1]} \quad (40)$$

In the limit of low $T \rightarrow 0$, we expect $\rho_{eff} \rightarrow 0$ so that the denominator may be approximated by keeping only the logarithm. This gives the following self-consistency equation.

$$\rho_{eff} \approx 4T\sqrt{K} \frac{\log(\frac{\sqrt{KT}}{\rho_{eff}})}{\log(\frac{4K\Lambda^2}{\rho_{eff}})} \quad (41)$$

This may be solved to give (in the low T limit)

$$\rho_{eff} \approx 4T\sqrt{K} \frac{\log(\log(\frac{1}{T}))}{\log(\frac{1}{T})} \quad (42)$$

Equation (42) shows that for small T , $\rho_{eff} \ll \sqrt{KT}$. This justifies our assumption in deriving equation(40). Also note that ρ_{eff} goes to zero as T goes to zero. Naive scaling based on the dynamical critical exponent $z = 2$ would have suggested $\rho_{eff} \sim T$. This is violated by logarithmic corrections which is exactly what is expected at the upper critical dimension due to the marginal irrelevance of the u term.

Now with this ρ_{eff} we have the following effective action for finite temperature:

$$S = \frac{T}{2} \int d^2x \{K(\nabla^2\chi)^2 + \rho_{eff}(\nabla\chi)^2\} \quad (43)$$

Therefore in this limit the equal time correlation function takes the form:

$$C_T(r) \approx \langle e^{i2\pi T(\chi(r) - \chi(0))} \rangle = e^{-\Phi_T(r)} \quad (44)$$

where

$$\Phi_T(r) = T \int d^2q \frac{1 - e^{iq \cdot r}}{Kq^4 + \rho_{eff}q^2} \quad (45)$$

Performing the angular integral above and rewriting in terms of the scaled variables $k = |q||r|$ and introducing the characteristic length scale ξ_T :

$$\xi_T^2 = K/\rho_{eff} \quad (46)$$

we have

$$\begin{aligned} \Phi_T(r) &= \frac{2\pi T r^2}{K} \int_0^\infty \frac{k dk (1 - J_0(k))}{k^4 + \frac{r^2}{\xi_T^2} k^2} \\ &= \frac{2\pi T \xi_T^2}{K} [K_0(r/\xi_T) + \log(r/\xi_T) + C - \log 2] \end{aligned} \quad (47)$$

where J_0 and K_0 are Bessel functions, and $C = 0.5772...$ is the Euler constant. The asymptotic properties of this function are as follows. First let us consider $r \ll \xi_T$:

$$\Phi_T(r \ll \xi_T) = \frac{\pi T r^2}{2K} [\log(2\xi_T/r) + (1 - C)] \quad (48)$$

This is essentially the correlation function we obtained in the scaling limit (32) with ξ_T playing the role of the system size and cutting off the divergent integral. This is as it should be; the length scale ξ_T represents the crossover scale from the scaling behaviour at shorter scales to the behaviour that is characteristic of the eventual finite temperature phase at larger scales. The behaviour at these larger scales can be obtained by studying the $r \gg \xi_T$ behaviour of the function above:

$$\Phi_T(r \gg \xi_T) = \frac{2\pi T}{\rho_{eff}} (\log(r/\xi_T) + C) \quad (49)$$

Thus, the form of the correlation function at finite temperatures at the longest scales is simply a power law $C_T(r) \sim 1/r^\sigma$ with an exponent $\sigma = 2\pi T/\rho_{eff}$ that diverges logarithmically as $T \rightarrow 0$. The divergence of the exponent at low- T implies a rapidly decaying power law form.

Several comments are in order about this result. First the correlation is a power law even at these finite temperature, because we have prohibited spinons in our theory (equivalently there are no defects in our height field χ). This will be remedied below by introducing gapped spinons. Second the form of the finite temperature spatial correlations is roughly consistent with the naive expectation that scaling should hold upto logarithmic corrections due to the marginally irrelevant operator. What is interesting however is that the important logarithmic correction occurs in the exponent σ of the power-law (equal-time) spatial correlation - in its absence the correlations are strictly local as shown explicitly in Section III A 2. We also note that the correlator above does not reproduce the zero temperature correlation function on simply substituting $T = 0$. This is only as expected as it arises entirely from the irrelevant u term. Note that while the scaling form for the autocorrelation function was already obtained in the scaling limit, one needs to include irrelevant operators to obtain finite spatial correlations. Thus a marked asymmetry in the origin of spatial and temporal correlations is evident.

It is useful to contrast the present model with other familiar models right at their upper critical dimension - for instance the $O(N)$ quantum critical point in three spatial dimensions. In all such cases it is necessary to include irrelevant operators in order to obtain the correct finite temperature correlations¹³. However in contrast to other critical theories at their upper critical dimension (eg. ϕ^4 theory), the quantum Lifshitz transition fixed point has operators with nontrivial scaling dimensions. The corresponding correlation functions might have been expected to show scaling at finite temperatures - but as we have seen the true behavior is more intricate.

2. Effect of Gapped Spinons

Now consider introducing spinons - i.e. vortex defects in the height field χ (these are absent in a pure quantum

dimer model, but are present in more physical representations of quantum magnets). Assume that these spinons have an energy gap E_c at the zero temperature quantum critical point. Then, in the finite temperature phase where an effective stiffness ρ_{eff} is generated, in addition to the core energy E_c , there is an additional contribution to the energy that is logarithmically divergent with the system size and takes the form:

$$E_v = \frac{\rho_{eff}}{4\pi} \log(L/a) \quad (50)$$

where L is the system size, a a microscopic lengthscale at which we may ascribe to the system an effective stiffness ρ_{eff} . Using the familiar Kosterlitz-Thouless criterion, we conclude that the entropy of vortex production, which is also logarithmically divergent with system size, wins over the energy cost if $\frac{\rho_{eff}}{4\pi} < 2T$. Since ρ_{eff}/T goes to zero as $T \rightarrow 0$, we conclude that the vortices (spinons) will be in the plasma phase, and the correlator (44) will eventually be an exponentially decaying function with a decay length set by $\xi_{spinon} \propto e^{-\frac{E_c}{2T}}$. Therefore, with a sufficiently large gap to spinons we have three regimes. First, for $r \ll \xi_T$ we have the quantum critical scaling regime with an effective system size cutoff set by ξ_T . Next, for $\xi_T \ll r \ll \xi_{spinon}$ we have power law correlators with a temperature dependent exponent σ . Finally, for $r \gg \xi_{spinon}$ we have an exponentially decaying function.

IV. PREDICTIONS FOR NUMERICAL EXPERIMENTS

Numerical experiments on quantum dimer models could directly verify the predictions of local criticality at these transitions. Perhaps the most readily accessible case for numerical experiments is the square lattice quantum dimer model with the RK hamiltonian. The zero temperature phase transition between a zero tilt and the staggered dimer phases that can be driven by varying parameters in the Hamiltonian is known to be a highly fine tuned version of the generic critical point discussed in^{2,3}, and considered in this paper. In particular it is known⁴ that the action (6) describes the asymptotic properties of the square lattice RK point with $K = \pi^2/4$ which may be compared against exact results¹². Correlators of the monopole insertion operator $V_r = \xi_r e^{i2\pi\chi_r}$ (where $\xi_r = \pm 1, \pm i$ is a Berry phase factor that oscillates on the four sublattices) correspond to correlators of the dimer bond/plaquette order. Since the bare RK point is a highly fine tuned critical point, it lacks a bare quartic term ($u = 0$), which can be seen from the absence of logarithmic factors in the exact expressions for correlation functions (which would otherwise arise if this marginally irrelevant operator were present). Finite temperature properties above the RK point are therefore expected to be as follows. The autocorrelation function of the monopole insertion operators should obey ω/T scaling, with the (scaling function part of the) spectral function given by equation (25) and plotted in Fig. III A 1.

The equal time correlators at spatially separated points though should approach zero in the scaling limit. Therefore the phenomenon of ‘local’ criticality should be visible in such numerical experiments. Non universal corrections to scaling, as calculated above on inclusion of the quartic term and gapped spinons, are not directly relevant to the pure RK point, since, as we noted before, it is a fine tuned point that lacks the quartic term, and the hard dimer constrain forbids spinons. Hence, the corrections to scaling will arise from the least irrelevant operator present, (e.g. the four monopole insertion operator that is non-oscillating and hence appears in the coarse grained action). A procedure similar to the one carried out here with the quartic term for the generic case, needs to be repeated with that operator to obtain the full asymptotics of the spatially separated correlators.

V. CONCLUSION

In this paper we have studied certain aspects of the finite temperature properties of the quantum Lifshitz transition discussed in Ref. 2,3,5. We first obtained exact information about the real time dynamical correlators at non-zero temperatures of certain important physical operators in the scaling limit. Such calculations are in general not possible for non-trivial quantum critical points in dimensions bigger than one. The quantum Lifshitz transition considered in this paper is a non-trivial quantum phase transition that nevertheless admits a free Gaussian description - this enables the calculation of the finite temperature dynamics. One of the remarkable results of this calculation is that in the scaling limit the correlators of the operators considered are strictly local in space though they are non-trivial power laws in time. This peculiar feature holds in the thermodynamic limit at non-zero temperatures. On the other hand if the temperature is allowed to go to zero first, and the thermodynamic limit taken later, spatial dependences indeed arise even in the scaling limit. Thus these quantum transitions provide an explicit example of a certain kind of ‘local’ behavior at a quantum critical point. However we emphasize that this is strictly a property of the model at non-zero temperature. The zero temperature fixed point is described by a fairly ordinary looking field theory. Spatial dependence of the correlators at non-zero temperature is restored once operators that are formally irrelevant at the fixed point are included. These corrections to scaling were calculated for two different classes of irrelevant perturbations, the quartic operator and gapped spinons, in Section III B.

What lessons may we learn for other quantum critical points? First the ‘local’ structure of the finite temperature scaling found is presumably special to this quantum Lifshitz transition - at least within the class of bosonic quantum critical points that are understood the best. It depends in part on realizing the special circumstance of a theory at its upper critical dimension that nevertheless

has operators with nontrivial scaling dimensions. Therefore, we expect that similar phenomena will *not* arise even at the other non-trivial deconfined critical points studied in Ref. 1. However in more complex situations with fermionic degrees of freedom, quantum phase transition phenomena are much less understood theoretically. The idea that some kind of spatial locality may be associated with the quantum critical fluctuations has been proposed at a phenomenological level to understand experiments in a few materials (such as the cuprates or heavy fermions where gapless fermionic excitations are undoubtedly present). Unfortunately it has thus far not been possible to develop any serious theoretical foundation for such ideas. Most of these proposals have assigned the locality observed in the finite temperature quantum critical region, to the local character of the zero temperature fixed point. In contrast, the zero temperature fixed point studied in this paper is *not* local in any sense. Rather, the local structure of correlations only arises when we consider the thermodynamic limit of the finite

temperature system (and we ignore all irrelevant operators). Hence the physics described in this paper provides a concrete example of a possible alternate route to some kind of ‘local’ criticality. It is hoped that the mathematical structure of the model studied in this paper might help in the search for similar phenomena in other models of strongly correlated system.

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 - ¹⁴ We regulate the sum over n with a factor $e^{-\frac{n}{n_c}}$, to eliminate the logarithmic divergence, and finally take $n_c \rightarrow \infty$.
 - ¹⁵ Note, that although the Fourier transform of equation (18) requires a short time cutoff to be defined for $\eta > 1$, the expression for the spectral function (23) is valid for any $\eta > 0$ and is independent of this cutoff.